Stereographic Projection

The Riemann Sphere and the Chordal Metric

Lecture Notes

September 15, 2026

1 Geometric Construction

Stereographic projection maps the coordinate grid from the complex plane $\mathbb{C}$ onto the unit sphere $S \subset \mathbb{R}^3$. The sphere is centered at the origin, with $\mathbb{C}$ identified with its equatorial plane $X_3 = 0$; the projection is defined by drawing a straight line from the North Pole $N = (0, 0, 1)$ of the sphere to a point on the complex plane, and the intersection of this line with the sphere defines the corresponding point on the sphere.

Interactive illustrations. `https://www.geogebra.org/m/PVSq4f7Q` and `https://www.geogebra.org/m/mkS2XdRa`.

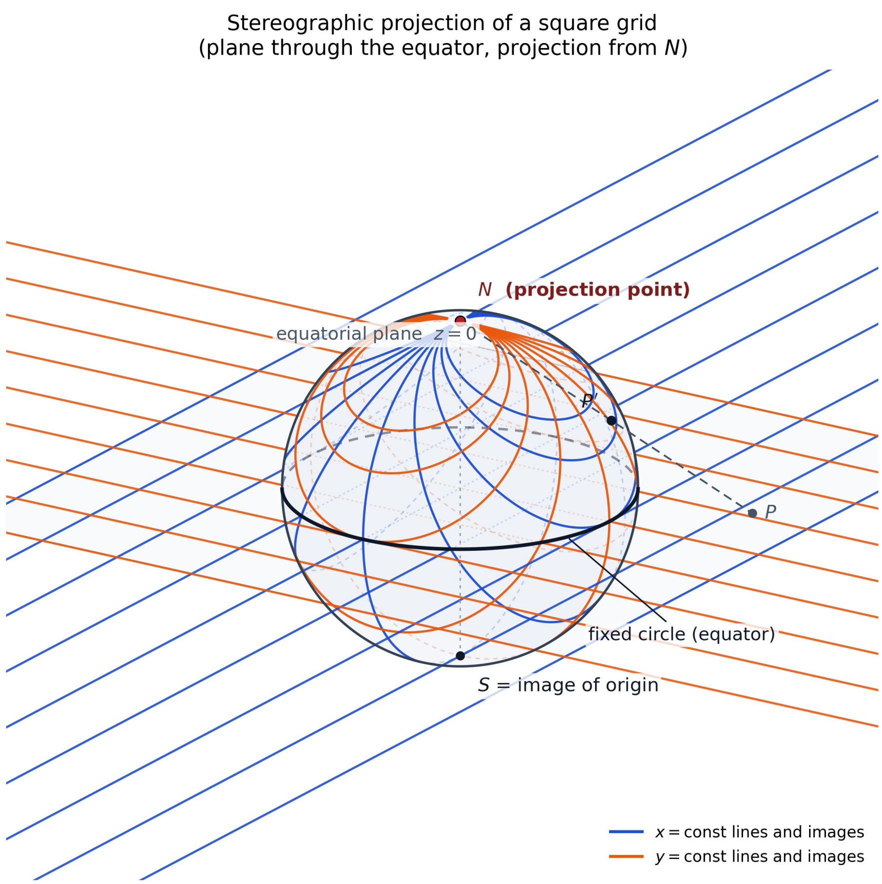

Figure 1: Stereographic projection of the coordinate grid to the sphere.

The North Pole N itself has no preimage in $\mathbb{C}$ under this projection. To make the correspondence a genuine bijection, we adjoin a single **point at infinity** to $\mathbb{C}$: write $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for the **extended**

complex plane, and identify ∞ with N. Under this identification, stereographic projection becomes a bijection $\hat{\mathbb{C}} \to S$, and S (equivalently, $\hat{\mathbb{C}}$ with this geometry) is called the **Riemann sphere**.

2 Algebraic Formulas

Let $z = x + iy$, and let (X_1, X_2, X_3) be the coordinates of its projection on the unit sphere. Using similar triangles formed by the North Pole, the point on the sphere, and the point on the plane, we get an algebraic relationship between them.

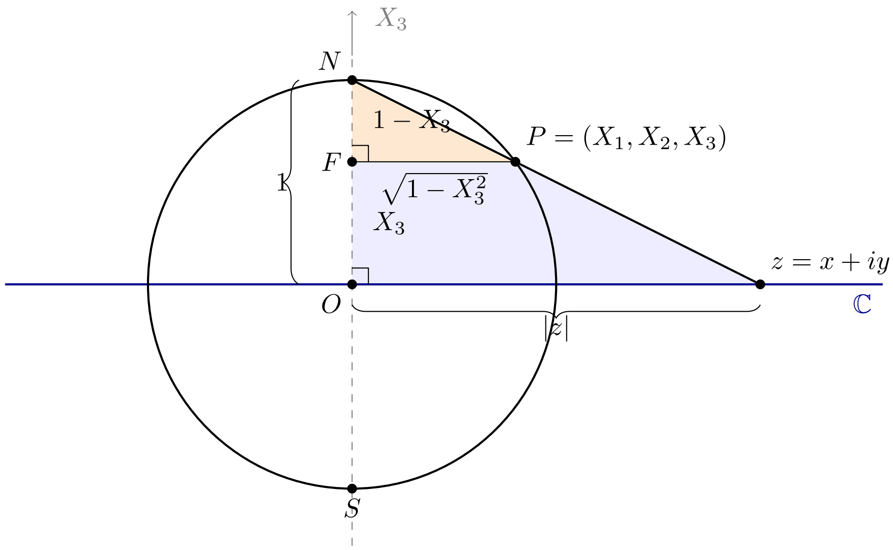

Figure 2: Stereographic projection from the north pole N onto the complex plane $\mathbb{C}$ (the equatorial plane $X_3 = 0$). The point z and its spherical image $P = (X_1, X_2, X_3)$ lie on the ray through N, and F is the foot of P on the axis. The right triangles $\triangle NOZ$ and $\triangle NFP$ are similar (their horizontal legs OZ and FP are parallel), so $\dfrac{\sqrt{1 - X_3^2}}{1 - X_3} = \dfrac{|z|}{1}$, whence $|z|^2 = \dfrac{1 - X_3^2}{(1 - X_3)^2} = \dfrac{1 + X_3}{1 - X_3}$ and $X_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$. Applying the same ratio to the in-plane components gives $X_1 = (1 - X_3)x = \dfrac{2\,\mathrm{Re}\, z}{|z|^2 + 1}$, $X_2 = (1 - X_3)y = \dfrac{2\,\mathrm{Im}\, z}{|z|^2 + 1}$, with inverse $z = \dfrac{X_1 + iX_2}{1 - X_3}$.

The projection formulas

$$X_3 = \frac{|z|^2 - 1}{|z|^2 + 1}, \qquad X_1 = (1 - X_3)x = \frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \qquad X_2 = (1 - X_3)y = \frac{2\,\mathrm{Im}(z)}{|z|^2 + 1},$$

with inverse

$$z = \frac{X_1 + iX_2}{1 - X_3}.$$

See Figure **??** for the derivation, obtained from

$$|z|^2 = \frac{1 - X_3^2}{(1 - X_3)^2} = \frac{1 + X_3}{1 - X_3}$$

by solving for X_3 and applying the same ratio to the in-plane components.

3 Spherical Distance (Chordal Metric)

The distance between two points Z, Z' on the sphere is the Euclidean distance $d(Z, Z') := |Z - Z'|$ in $\mathbb{R}^3$. By geometric similarity and the Pythagorean theorem, the distance from the North Pole to a projected point is

$$|N - Z| = \frac{2}{\sqrt{1 + |z|^2}}.$$

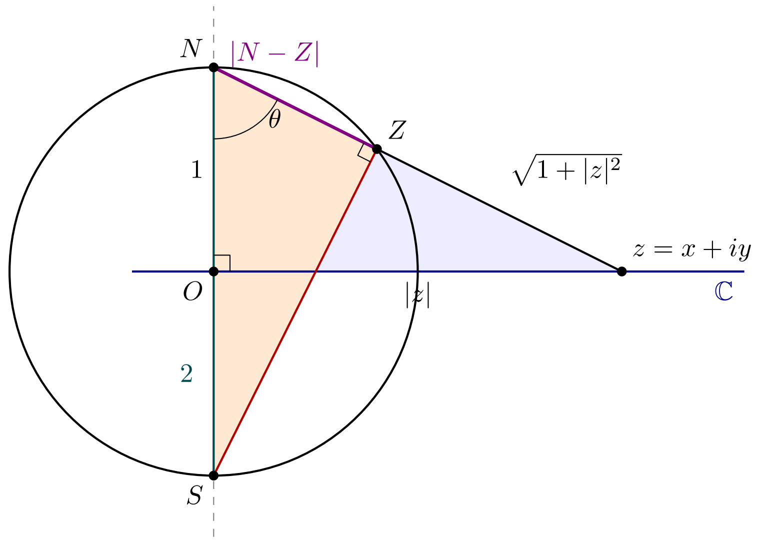

Figure 3: Deriving $|N - Z|$. The right triangles $\triangle NOz$ and $\triangle NZS$ share the angle θ at N (both Z and z lie on the same ray from N), and are right-angled at O and at Z respectively – the latter by Thales' theorem, since NS is a diameter of S. Hence they are similar, so $\frac{NZ}{NS} = \frac{NO}{Nz}$, i.e. $\frac{|N-Z|}{2} = \frac{1}{\sqrt{1+|z|^2}}$, giving $|N - Z| = \frac{2}{\sqrt{1+|z|^2}}$.

Because the triangles $\triangle NZZ'$ and $\triangle Nzz'$ are similar, this yields the chordal metric.

Theorem (chordal metric)

$$d(z, z') = \frac{2|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}, \qquad d(z, \infty) = \lim_{z' \to \infty} d(z, z') = \frac{2}{\sqrt{1 + |z|^2}}.$$

See Figure **??**.

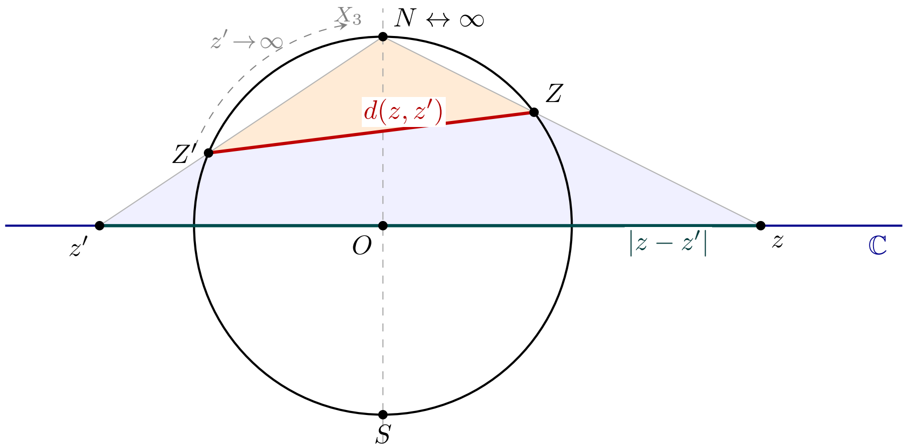

Figure 4: The chordal (spherical) metric. Points $z, z' \in \mathbb{C}$ project from the north pole N to Z, Z' on the sphere, and the chordal distance is the Euclidean chord $d(z, z') = |Z - Z'|$. Along each projection ray the product of the two distances from N is constant: $NZ \cdot Nz = NZ' \cdot Nz' = 2$, since $|N - Z| = \frac{2}{\sqrt{1+|z|^2}}$ and $|N - z| = \sqrt{1 + |z|^2}$. Hence $\triangle NZZ' \sim \triangle Nz'z$ (common angle at N, enclosing sides in proportion), and the ratio of corresponding sides gives $d(z, z') = \dfrac{2\,|z - z'|}{\sqrt{1+|z|^2}\,\sqrt{1+|z'|^2}}$. Since N corresponds to the point at infinity, letting $z' \to \infty$ slides $Z' \to N$, whence $d(z, \infty) = |N - Z| = \dfrac{2}{\sqrt{1+|z|^2}}$.

4 Circles and Straight Lines

A fundamental property of stereographic projection is that circles and straight lines in $\mathbb{C}$ are always mapped to circles on the sphere.

Straight lines. The image of a line in the plane is the intersection of the sphere S with the plane through the line and the North Pole N – a circle passing through N.

Circles. A circle $(x - a)^2 + (y - b)^2 = r^2$ in the plane, under the projection formulas, becomes the equation of a plane:

$$aX_1 + bX_2 + \frac{a^2 + b^2 - 1 - r^2}{2} X_3 = \frac{a^2 + b^2 - r^2 + 1}{2}.$$

So the image is the intersection of this plane with S. Conversely, any plane intersecting the sphere but not passing through N corresponds to a circle in $\mathbb{C}$.

Proof of the converse. Write a general plane as $AX_1 + BX_2 + CX_3 = D$; since it does not pass through $N = (0, 0, 1)$, we have $C \neq D$. Substituting the projection formulas and writing $z = x + iy$, $X_1 = \dfrac{2x}{|z|^2+1}$, $X_2 = \dfrac{2y}{|z|^2+1}$, $X_3 = \dfrac{|z|^2-1}{|z|^2+1}$ and clearing the denominator $|z|^2 + 1$ turns the plane equation into

$$2Ax + 2By + C(|z|^2 - 1) = D(|z|^2 + 1),$$

i.e.

$$(C - D)|z|^2 + 2Ax + 2By = C + D.$$

Since $C \neq D$, dividing by $C - D$ gives

$$x^2 + y^2 + \frac{2A}{C-D}\,x + \frac{2B}{C-D}\,y = \frac{C+D}{C-D},$$

which is the equation $(x-\alpha)^2 + (y-\beta)^2 = \rho^2$ of a circle in $\mathbb{C}$, with center $\alpha = -\dfrac{A}{C-D}$, $\beta = -\dfrac{B}{C-D}$, and $\rho^2 = \alpha^2 + \beta^2 + \dfrac{C+D}{C-D}$ (positive precisely because the plane meets the sphere in a circle, not a single tangent point). □

The excluded case. When $C = D$ (i.e. the plane passes through N), the same substitution gives $2Ax + 2By = C + D$ with no $|z|^2$ term – a *line* in $\mathbb{C}$, not a circle. This is exactly the converse of the "Straight lines" case above: a plane through N corresponds to a line in $\mathbb{C}$.

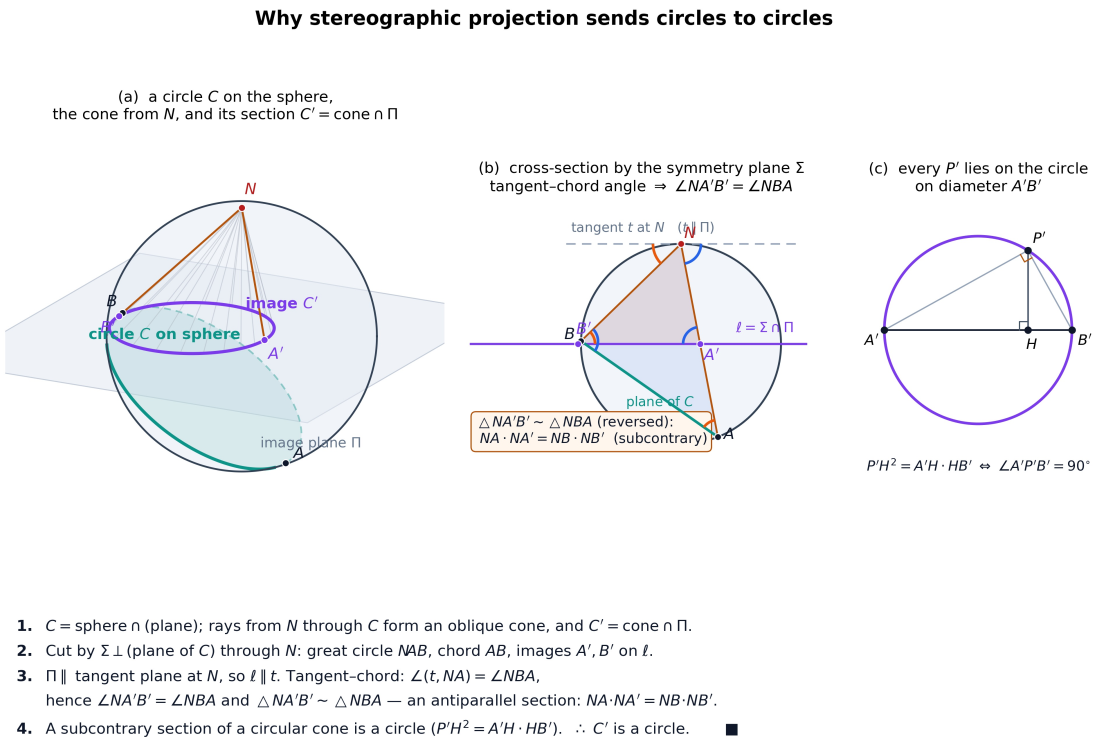

Figure 5: Geometric proof that circles on the sphere project to circles in the plane.

See Figure ??, and `https://www.youtube.com/watch?v=QqxNpYQYTFA` for a geometric proof.

5 Transformations and Isometries

Certain transformations of $\mathbb{C}$ correspond to coordinate reflections of the sphere S and preserve the spherical distance $d(z, z')$:

- **Conjugation** $z \mapsto \bar{z}$ corresponds to $(X_1, X_2, X_3) \mapsto (X_1, -X_2, X_3)$.

  Derivation. Writing $z = x + iy$, $\bar{z} = x - iy$ has the same modulus $|z|$, so $X_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$ is unchanged; in the formulas for X_1, X_2, replacing (x, y) by $(x, -y)$ negates only $X_2 = \dfrac{2y}{|z|^2 + 1}$.

- $z \mapsto 1/\bar{z}$ corresponds to $(X_1, X_2, X_3) \mapsto (X_1, X_2, -X_3)$.

  Derivation. $1/\bar{z} = z/|z|^2$, so $|1/\bar{z}| = 1/|z|$. Substituting $|z| \mapsto 1/|z|$ into $X_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$ gives

$\dfrac{1/|z|^2-1}{1/|z|^2+1} = \dfrac{1-|z|^2}{1+|z|^2} = -X_3$, while $\mathrm{Re}(1/\bar{z}) = \mathrm{Re}(z)/|z|^2$ and $|1/\bar{z}|^2+1 = (1+|z|^2)/|z|^2$ combine to leave $X_1 = \dfrac{2\,\mathrm{Re}(z)}{|z|^2+1}$ unchanged, and likewise for X_2.

- **Inversion** $z \mapsto 1/z$ corresponds to $(X_1, X_2, X_3) \mapsto (X_1, -X_2, -X_3)$.
  Derivation. Since $1/z = \overline{1/\bar{z}}$, inversion is the composition of $z \mapsto 1/\bar{z}$ (which negates X_3) with conjugation (which negates X_2); applying both gives $(X_1, X_2, X_3) \mapsto (X_1, -X_2, -X_3)$.

All of these preserve $d(z, z')$: each corresponds to an orthogonal reflection of $\mathbb{R}^3$ (negating one or two of the coordinates X_1, X_2, X_3), and an orthogonal map preserves Euclidean distance, hence preserves $d(Z, Z') = |Z - Z'|$ by definition.

However, translations such as $z \mapsto z+1$ do *not* preserve the spherical distance: for instance $d(0, \infty) = 2$ but $d(1, \infty) = \dfrac{2}{\sqrt{2}} = \sqrt{2} \neq 2$.